The number of small covers over cubes

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Outline

Toric Topology

What is Toric Topology?

Toric Manifold

- Toric Variety : a normal complex algebraic variety with $(\mathbb{C}^*)^n$ action having a dense orbit.
- Toric manifold : a compact smooth toric variety.

- Quasitoric Manifold : a closed smooth manifold M of dim. $2n$ with a smooth $(S^1)^n$ action such that
  - the action is *locally standard*,
  - the orbit space $M/(S^1)^n$ is a simple convex polytope.
- Torus Manifold : a closed smooth orientable manifold of dim. $2n$ with a smooth effective $(S^1)^n$ action having a fixed point.

- Small cover : a closed smooth manifold M of dim. n with a smooth $(\mathbb{Z}_2)^n$ action such that
  - the action is *locally standard*,
  - the orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.
- 2-Torus Manifold : a closed smooth orientable manifold of dim. n with a smooth effective $(\mathbb{Z}_2)^n$ action having a fixed point.

- P : simple polytope of dim n.
- $\mathfrak{F}(P) = \{F_1, \ldots, F_m\}$: the set of facets of P.
- $T = \mathbb{Z}_2^n$: real torus of dim. n.
- $\lambda : \mathfrak{F}(P) \to H_2(BT) = \text{Hom}(\mathbb{Z}_2, T) = \mathbb{Z}_2^n$: Characteristic Function if

$$\cap F_i \text{ is a vertex} \Rightarrow \{\lambda(F_i)\} \text{ is a basis of } \mathbb{Z}_2^n$$

This condition is called the non-singularity condition

Construction

- $F = \cap_j F_j$: face of P.
- $T_F \subset T$: the torus subgroup generated by $\lambda(F_j)$.

$$M(\lambda) = P \times T / \sim$$

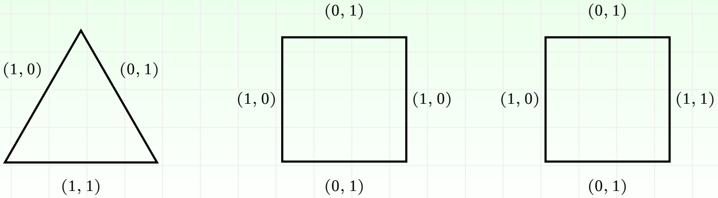
Here

$$(p,g) \sim (q,h) \Leftrightarrow p = q \text{ and } g^{-1}h \in T_{F(p)}$$

where $F(p)$ is the face which contains p in its interior.

- A small cover is a smooth closed manifold M^n with
  - a locally standard $\mathbb{Z}_2^n$ action
  - its orbit space is a simple convex polytope
- $M(\lambda)$ is a small cover over P.

Examples



- $\mathbb{R}P^2 = \{(z_0 : z_1 : z_2), z_i \in \mathbb{R}\}$ is a small cover with $\mathbb{Z}_2^2$-action $(t_1, t_2) \cdot (z_0 : z_1 : z_2) = (z_0 : t_1 z_1 : t_2 z_2)$. Then $\mathbb{R}P^2/\mathbb{Z}_2^2 = \Delta^2$
- The second is T^2 with $\mathbb{Z}_2^2$ acting as a reflection group
- The third is one is the Klein bottle and is equivariantly homeomorphic to $\mathbb{R}P^2 \sharp \mathbb{R}P^2$

- P : Simple convex polytope of dim n.
- $cf(P)$: Set of all characteristic functions over P.

M.Davis and T.Januszkiewicz

All small covers over P are given by $\{M(\lambda) | \lambda \in cf(P)\}$.

- M_1, M_2 : small covers over P.
- M_1 and M_2 are equivalent up to ... :
  - Equivariant homeo. (or $\mathbb{Z}_2^n$ - homeo) if

    $$\exists f : M_1 \longrightarrow M_2 \text{ s.t. } f(t \cdot x) = t \cdot f(x)$$

  - Weakly equivariant homeo. (or weak $\mathbb{Z}_2^n$ - homeo) if

    $$\exists f : M_1 \longrightarrow M_2 \text{ s.t. } f(t \cdot x) = \varphi(t) \cdot f(x),$$

    where $\varphi \in \mathrm{Aut}(\mathbb{Z}_2^n)$.
  - D-J equivalence if $\exists f$: weakly equivariant homeo. and f covers identity on P.

History

A.Garrison and R.Scott (2002)

- They give us the computer algorithm for counting the number of D-J equivalent classes.
- P: dodecahedron,
  - $\sharp DJ(P) = 2165$
  - $\sharp homeo(P) = 25$

M.Cai, X.Chen and Z.Lu (2007)

- P : 3-dimensional prism. i.e. $P =$ polygon $\times I$.
  - $\sharp DJ(P)$
  - $\sharp Z_2^n - diff(P)$

Outline

Introduction

Small covers over cubes

Orientable small covers over cubes

Equivariant homeomorphism classes

Homeomorphism classes

The purposes of this talk are

- Counting $\sharp DJ(I^n)$ and $\sharp DJ(\prod_{i=1}^{m} \Delta^{n_i})$.
- Counting orientable small covers over I^n
- Counting $\sharp \mathbb{Z}_2^n - homeo(I^n)$.
- Upper bound of the $\sharp homeo(I^n)$.

Why study small covers over cubes?

1. Nice connection between linear algebra and topology
2. Small covers over cubes are flat manifolds. (from the viewpoint of differential geometry).
3. By this work, connection with graph theory.

One may assign an $(n \times m)$-matrix Λ to an element $\lambda \in cf(P)$ by ordering the facets and choosing a basis for $\mathbb{Z}_2^n$.

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_n)) = (A|B),$$

where A be a $(n \times n)$-matrix and B be a $(n \times (m-n))$-matrix. Since there is 1-1 correspondence

$$\{\text{D-J classes over } P\} \leftrightarrow \text{GL}(n, \mathbb{Z}_2) \setminus cf(P),$$

up to D-J equivalence, $\Lambda \sim (E_n | A^{-1}B)$.
Denote Λ_*=$A^{-1}B$.

When $P = I^n$,

- $\sharp$ of facets = $2n \Rightarrow \Lambda$ is $(n \times 2n)$-matrix, i.e. Λ_* is $(n \times n)$ matrix.
- Labeling each facets satisfying $F_i \cap F_{n+i} = \phi$.
- non-singularity condition $\Leftrightarrow$ Every principal minor of Λ_* is 1.

Let $m(n)$ be a set of matrices all of whose principal minor is 1.

$$\{\text{D-J classes over } I^n\} \leftrightarrow m(I^n)$$

Acyclic digraph

- Digraph : Graph with at most one edge directed from vertex i to vertex j, for $1 \leq i \leq n, 1 \leq j \leq n$.
- Acyclic : Graph without cycles of any length.

- $\mathfrak{G}_n$: the set of acyclic digraphs with n labeled nodes.
- $A(G)$: The vertex adjacency matrix of $G \in \mathfrak{G}_n$.

Theorem

Theorem

There is a bijection $\phi : \mathfrak{G}_n \to m(n)$ *by*

$$\phi : G \mapsto A(G) + E_n$$

where E_n *is an identity matrix of size* n*.*

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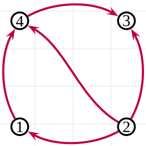

$$\xrightarrow{\phi} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Conjugation action

- σ : Permutation on $\{1, 2, \ldots, n\}$.
- $P(\sigma) = (p_{ij})$: corresponding $(n \times n)$ permutation matrix, where

$$p_{ij} = \begin{cases} 1, & \text{if } (i,j) = (\sigma(j), j) \\ 0, & \text{otherwise.} \end{cases}$$

- Conjugation action on $(n \times n)$-matrices

$$A \mapsto P(\sigma)^{-1} A P(\sigma)$$

Conjugation action on $m(n) \overset{1:1}{\longleftrightarrow}$ Relabelling Nodes of $\mathfrak{S}_n$

Sketch of Proof

Lemma

Let A be an $n \times n$ $\mathbb{Z}_2$-matrix all of whose proper principal minor is 1. Then

$$\exists P \text{ s.t } P^{-1}AP = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

Sketch of Proof

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Relabeling preserves acylicness of graph!!

□

R.Robinson(1970) and R.Stanley(1973)

Let R_n be the number of acyclic digraphs with n labeled nodes.

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}$$

n	0	1	2	3	4	5	6	7	$\cdots$
R_n	1	1	3	25	543	29281	3781503	1138779265	$\cdots$

Over a product of simplices

Theorem

$$\sharp DJ\left(\prod_{i=1}^{\ell}\Delta^{n_i}\right) = \sum_{G\in\mathfrak{G}_\ell}\prod_{v_i\in G}(2^{n_i}-1)^{\mathit{outdeg}(v_i)}$$

Over a product of simplices

Theorem

$$\sharp DJ(\prod_{i=1}^{\ell} \Delta^{n_i}) = \sum_{G \in \mathfrak{G}_\ell} \prod_{v_i \in G} (2^{n_i} - 1)^{outdeg(v_i)}$$

$\underline{m=2}$: $\sharp DJ(\Delta^{n_1} \times \Delta^{n_2}) = 1 + (2^{n_1} - 1) + (2^{n_2} - 1)$.
$\underline{m=3}$:
$\sharp DJ(\Delta^{n_1} \times \Delta^{n_2} \times \Delta^{n_3}) = 1 + 2(x_1 + x_2 + x_3) + (x_1 + x_2 + x_3)^2 + (x_1 x_2 + x_2 x_3 + x_3 x_1) + (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) - x_1^3 - x_2^3 - x_3^3$,
where $x_i = 2^{n_i} - 1$ for $i = 1, 2, 3$.

Outline

H.Nakayama and Y.Nishimura (2005)

$$M(\lambda) \text{ is orientable } \iff \exists \text{ a basis of } \mathbb{Z}_2^n \text{ s.t. } Im(\epsilon\lambda) = \{1\}$$

where $\epsilon : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is defined by $\epsilon(e_i) = 1$ for all i.

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Corollary

The number of orientable small covers over I^n = The number of acyclic digraphs with labeled nodes of even out-degree.

Let O_n be the number of orientable small covers over I^n.

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(k-1)(n-k)} R_{n-k}$$

n	0	1	2	3	4	5	6	7	$\cdots$
O_n	1	1	1	4	43	1156	74581	11226874	$\cdots$

Theorem

$$\frac{O_n}{R_n} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Analysis

- $F(x) = \sum \frac{x^n}{n!2^{\binom{n}{2}}}$
- $R(x) = \sum R_n \frac{x^n}{n!2^{\binom{n}{2}}}$, $O(x) = \sum O_n \frac{x^n}{n!2^{\binom{n}{2}}}$

R.Stanley(1973)

$$R(x)F(-x) = 1$$

Corollary

$$O(x)F(-\frac{x}{2}) = 1 - F(-x)$$

Problem

$$\frac{O_{n+1}}{R_{n+1}} / \frac{O_n}{R_n} \to \frac{1}{2} \text{ as } n \to \infty.$$

Outline

- Aut$(P) : \mathfrak{F}(P) \to \mathfrak{F}(P)$ preserving the face poset structure.

(Known) For $\lambda_1, \lambda_2 \in cf(P)$,

$M(\lambda_1) \approx M(\lambda_2)$up to equiv. homeo. $\Leftrightarrow \exists h \in$ Aut(P) s.t. $\lambda_1 = \lambda_2 \circ h$

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Burnside's formula G : finite group action on a set X.

$$\sharp \text{ of orbits } = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where $X^g = \{x \in X | gx = x\}$.

Set P be a cube I^n. All automorphisms of $\mathrm{Aut}(I^n) \curvearrowright cf(I^n)$ can be written in a simple form as follows :

$$\mu \cdot \chi_1^{\varepsilon_1} \cdot \dots \cdot \chi_n^{\varepsilon_n}, \varepsilon_j \in \mathbb{Z}_2$$

with $\mu \in S_n$ and $\chi_1^2 = \cdots = \chi_n^2 = 1$.

- $|\mathrm{Aut}(I^n)| = 2^n n!$.
- $|X^{\chi_1 \cdot \dots \cdot \chi_k}| = 2^{k(n-k)} R_k$.
- $|X^{\mu \cdot \chi_1^{\varepsilon_1} \cdot \dots \cdot \chi_n^{\varepsilon_n}}| = 0$ if $\mu \neq 1$
- $|cf(I^n)| = |m(n)| \times |GL(n, \mathbb{Z}_2)|$ since $GL(n, \mathbb{Z}_2) \underset{\text{free}}{\curvearrowright} cf(I^n)$

Theorem

Let Q_n be a number of equivariant homeomorphism classes of small covers over I^n.

Theorem

$$Q_n = \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} R_k}{2^n n!} \cdot \prod_{i=0}^{n-1} (2^n - 2^i)$$

n	0	1	2	3	4	5	$\cdots$
Q_n	1	1	6	259	87360	236240088	$\cdots$

Outline

Let T_n be the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over I^n.

$$T_n = |GL(n, \mathbb{Z}_2) \setminus cf(I^n)/\mathrm{Aut}(I^n)| \leq |m(n)/\sim|$$

where $\sim$ is equivalent class by conjugate action.

Theorem

T_n is less than or equal to the number of acyclic digraphs with n unlabeled nodes.

Corollary

The number H_n of homeomorphism classes of small covers over I^n is less than or equal to the number of acyclic digraphs with n unlabeled nodes.

n	0	1	2	3	4	5	6	7	$\cdots$
$H_n \leq$	1	1	2	6	31	302	5984	243668	$\cdots$

M.Masuda (Unpublished yet)

n	0	1	2	3	4
H_n	1	1	2	5	12

Thank you for listening!